ON THE METHOD OF CONTINUED IDENTITY.*

By Mr. Chas. H. Kummell, Washington, D. C.

This apparently new method of expressing a quantity is based on the following considerations:—

Suppose a quantity x is approximately known by its functional properties, so that we can place

$$x = \frac{1 + x_1}{q}, \tag{1}$$

where $1/q$ is the approximate value of x and x_1/q its correction. Again, assume

$$x_1 = qx - 1 = \frac{1 + x_2}{q_1}, \tag{1_1}$$

$$x_2 = q_1 x_1 - 1 = \frac{1 + x_3}{q_2}, \tag{1_2}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$x_n = q_{n-1} x_{n-1} - 1 = \frac{1 + x_{n+1}}{q_n}; \tag{1_n}$$

then we shall have

$$x = \frac{1}{q}\left[1 + \frac{1}{q_1}\left[1 + \frac{1}{q_2}\left[1 + \ldots\right.\right.\right. \tag{2}$$

$$= \frac{x}{1 + x_1}\left[1 + \frac{x_1}{1 + x_2}\left[1 + \frac{x_2}{1 + x_3}\left[1 + \ldots\right.\right.\right. \tag{3}$$

Suppose any one x, for instance x_3, $= 0$, or sufficiently nearly so; then we have

$$\begin{aligned} x &= \frac{x}{1 + x_1}\left[1 + \frac{x_1}{1 + x_2}(1 + x_2)\right] \\ &= \frac{x}{1 + x_1}(1 + x_1) \\ &= x, \quad \text{a perfect identity;} \end{aligned}$$

and it is important to notice that this identity holds, whatever the preceding values of x_2 and x_1 may have been, if only $x_3 = 0$ or is sufficiently small to be

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neglected. We may thus continue the identity to any number of steps before perfect or sufficiently perfect identity is reached. Since

$$q = \frac{1 + x_1}{x} = \frac{1}{x}, \quad \text{approximately,}$$

$$q_1 = \frac{1 + x_2}{x_1} = \frac{1}{x_1} \quad \text{``}$$

$$q_2 = \frac{1 + x_3}{x_2} = \frac{1}{x_2} \quad \text{``}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$q_n = \frac{1 + x_{n+1}}{x_n} = \frac{1}{x_n} \quad \text{``}$$

the condition for perfect identity of (2) is that the q's must have ∞ for their limit. Take, for instance, the binomial series which may be written

$$(1 + x)^m = 1 + \frac{m}{1} x \left[1 + \frac{m-1}{2} x \left[1 + \frac{m-2}{3} x \left[1 + \ldots ; \right.\right.\right. \tag{4}$$

comparing which with (2), we have

$$q = 1, \tag{5}$$

$$q_1 = \frac{1}{mx}, \tag{5_1}$$

$$q_2 = \frac{2}{(m-1)x}, \tag{5_2}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$q_n = \frac{n}{(m-n+1)x}. \tag{5_n}$$

The condition for perfect identity is then

$$q_n = \infty; \qquad \therefore m - n + 1 = 0.$$

It follows, then, that the binomial series can give perfect identity only if m is an integer, and that this is attained by $m + 1$ terms. If m is not an integer, perfect identity is impossible, since q_n can never be infinite; and this is also the case if the condition of convergence $x < 1$ is fulfilled. Since, then,

$$q_\infty = \left. \frac{n}{(m-n+1)x} \right]_{n=\infty} = -\frac{1}{x} > -1,$$

the terms at infinity form a geometric series with the exponent $-x$, which therefore has a sum. It follows, then, that the form of continued identity is

still convergent if the denominators q tend to a limit > 1. This condition holding, we can give an approximate value of the remainder if we stop at any term; for if q_n is sufficiently large, the next term, though small, will exceed the sum of all following terms; and since q_{n+1} must be at least $= q_n$, we have

$$(n+1)\text{th term} < \frac{1}{q} \cdot \frac{1}{q_1} \cdot \frac{1}{q_2} \cdot \cdot \cdot \frac{1}{q_n} \cdot \frac{1}{q_n} = \text{approximate remainder.} \qquad (6)$$

Transforming any equation for one unknown quantity x by relations (1) into an equation for $x_1, x_2, \ldots$, we have a method of approximation which is convenient and, in general, rapidly convergent. Suppose we have to solve

$$0 = f(x). \qquad (7)$$

Let

$$x = x_0 + x_1, \qquad (8)$$

where x_0 is the integral and x_1 the fractional part of the root. By Taylor's theorem we have

$$0 = f(x_0) + f'(x_0)\,x_1 + \frac{f''(x_0)}{2!}\,x_1^2 + \frac{f'''(x_0)}{3!}\,x_1^3 + \ldots \qquad (9)$$

Let

$$x_1 = \frac{1 + x_2}{q_1}, \qquad (1_1)$$

where q is the integral part of $-\dfrac{f'(x_0)}{f(x_0)}$ (preferably exceeding it). To transform (9) to an equation in x_2 we have to decide first to which order of precision we wish to satisfy the equation. To fix ideas, let us suppose the solution to be exact to the 3d order; we have then to solve a cubic equation, which we shall write thus:

$$0 = a_1''' + a_1''x_1 + a_1'x_1^2 + a_1x_1^3. \qquad (9_1)$$

Transforming this to one having its roots q_1 times as great, we have

$$0 = q_1^3a_1''' + q_1^2a_1''(q_1x_1) + q_1a_1'(q_1x_1)^2 + a_1(q_1x_1)^3;$$

then transforming this to one having its roots diminished by 1, we have

$$\begin{aligned} 0 &= a_1 + a_1'q_1 + a_1''q_1^2 + a_1'''q^3 \\ &\qquad + (3a_1 + 2a_1'q_1 + a_1''q_1^2)\,x_2 + (3a_1 + a_1'q_1)\,x_2^2 + a_1x_2^3 \\ &= a_2''' + a_2''x_2 + a_2'x_2^2 + a_2x_2^3. \end{aligned} \qquad (9_2)$$

Let

$$x_2 = \frac{1 + x_3}{q_2}, \qquad (1_2)$$

where q_2 = integral part of $-\frac{a_2''}{a_2'''}$ (preferably exceeding it). Transforming to $q_2x_2 - 1$ as before, we have

$$0 = a_3''' + a_3''x_3 + a_3'x_3^2 + a_3x_3^3.$$

This process is continued as far as we please, in case the cubic (9_1) is proposed for solution. If, however, (9_1) coincides with the complete equation (9) to the 3d order only, it will be useless to make more than three transformations. A convenient form for making these transformations I shall exhibit in the solution of the cubic

$$0 = 3 - 5x + x^3.$$

This equation has three real roots, one between 0 and 1, another between 1 and 2, and the third between -2 and -3. Here $\frac{3}{5}$ is an approximate value for the smallest root; hence $q = 2 > \frac{3}{5}$, and a more exact value of the root is found by the following algorithm:

$$\begin{array}{rrrrrl}
0 = & +3 & -5x & +0x^2 & +x^3 & x = \dfrac{1+x_1}{2} \\
 & +6 & -10 & 0 & +1 & \\
 & +12 & -20 & +1 & +1 & \\
 & +24 & -19 & +2 & +1 & \\
0 = & +5 & -17x_1 & +3x_1^2 & +x_1^3 & x_1 = \dfrac{1+x_2}{4} \\
 & +20 & -68 & +12 & +1 & \\
 & +80 & -272 & +13 & +1 & \\
 & +320 & -259 & +14 & +1 & \\
0 = & +61 & -245x_2 & +15x_2^2 & +x_2^3 & x_2 = \dfrac{1+x_3}{4} \\
 & +244 & -980 & +60 & +1 & \\
 & +976 & -3920 & +61 & +1 & \\
 & +3904 & -3859 & +62 & +1 & \\
0 = & +45 & -3797x_3 & +63x_3^2 & +x_3^3 & x_3 = \dfrac{1+x_4}{84}
\end{array}$$

The root so far determined is correct within less than $\frac{1}{2 \times 4 \times 4 \times 84^2} = \frac{1}{86016}$. We have, then, neglecting x_4,

$$\begin{aligned}
x_3 &= 0.011905, \\
x_2 &= 0.252976, \\
x_1 &= 0.313244, \\
x &= 0.656622.
\end{aligned}$$

The correct value is $x = 0.656620.$

To determine another root, for instance that lying between -2 and -3, we may first increase the roots by 2 by assuming $x = y - 2$; then y will lie between 0 and -1, and the more exact value is found thus:

$$
\begin{array}{lrlrlrlr}
0 = + & 5 & + & 7y & - & 6y^2 & + y^3 & \qquad y = \dfrac{1 + y_1}{-2} \\
- & 10 & - & 14 & + & 12 & + 1 & \\
+ & 20 & + & 28 & + & 13 & + 1 & \\
- & 40 & + & 41 & + & 14 & + 1 & \\
0 = + & 1 & + & 55y_1 & + & 15y_1^{\,2} & + y_1^{\,3} & \qquad y_1 = \dfrac{1 + y_2}{-55} \\
- & 55 & - & 3025 & - & 825 & + 1 & \\
+ & 3025 & + & 166375 & - & 824 & + 1 & \\
- & 166375 & + & 165551 & - & 823 & + 1 & \\
0 = - & 824 & + & 164728y_2 & - & 822y_2^{\,2} & + y_2^{\,3} & \qquad y_2 = \dfrac{1 + y_3}{200}
\end{array}
$$

This determination must be correct within less than $\dfrac{1}{2 \times 55 \times 200^2} = \dfrac{1}{4400000}$, and we have

$$
\begin{aligned}
y_2 &= +\,0.005, \\
y_1 &= -\,0.0182727, \\
y &= -\,0.4908636, \\
x &= -\,2.4908636.
\end{aligned}
$$

The correct value is $\qquad x = -\,2.4908638.$

We notice the greater convergence in the computation of this root. This is caused by its being more different from the other roots.

Shculd two or more roots be very close together, the process becomes very slowly convergent. This is, indeed, a difficulty common to all numerical methods of solution of higher equations. It may be in most cases sufficiently overcome by placing the nearly equal roots between 0 and 1, and then determine their reciprocals. For a test of this suggestion, I solve the equation,

$$0 = 7 - 7x + x^3,$$

which has two roots between $+1$ and $+2$. Here the direct process gives

$$
\begin{aligned}
x &= \frac{1}{1}\Big[1 + \frac{1}{4}\Big[1 + \frac{1}{3}\Big[1 + \frac{1}{3}\Big[1 - \frac{1}{7}\Big[1 + \frac{1}{16}\Big[1 + \cdots \\
&= 1.3568948.
\end{aligned}
$$

If we assume $x = y + 1$, then

$$0 = 1 - 4y + 3y^2 + y^3\,;$$

and placing $y = z^{-1}$, we have

$$0 = 1 + 3z - 4z^2 + z^3.$$

This equation has one positive root between $+1$ and $+2$, and one between $+2$ and $+3$. To determine the larger, which corresponds to the root already given, we assume $z = v + 3$, and have

$$\begin{array}{rrrrrrl}
0 = & +\;1 & +\;6v & +\;5v^2 & +\,v^3 & & v = \dfrac{1+v_1}{-6} \\
 & -\;6 & -\;36 & -\;30 & +\,1 & & \\
 & +\;36 & +\;216 & -\;29 & +\,1 & & \\
 & -\;216 & +\;187 & -\;28 & +\,1 & & \\
0 = & -\;29 & +\;159v_1 & -\;27v_1{}^2 & +\,v_1{}^3 & & v_1 = \dfrac{1+v_2}{5} \\
 & -\;145 & +\;795 & -\,135 & +\,1 & & \\
 & -\;725 & +\,3975 & -\,134 & +\,1 & & \\
 & -\,3625 & +\,3841 & -\,133 & +\,1 & & \\
0 = & +\;216 & +\,3708v_2 & -\,132v_2{}^2 & +\,v_2{}^3 & & v_2 = \dfrac{1+v_3}{-17}
\end{array}$$

This should give the root within less than $\dfrac{1}{6 \times 5 \times 17^2} = \dfrac{1}{8670}$, and we have

$$\begin{aligned}
v_2 &= -\,0.05882, \\
v_1 &= +\,0.18824, \\
v &= -\,0.19804, \\
z &= +\,2.80196, \\
y &= +\,0.35689, \\
x &= +\,1.35689.
\end{aligned}$$

The exact value is $\quad x = +\,1.356896.$

If we apply the algorithm to the equation

$$0 = 8 - 7x + x^3,$$

which differs from the above equation by 1 in the numerical term, the two roots corresponding to those before nearly equal will be imaginary, and we have

$$
\begin{array}{rrrrrl}
0 = & +\ 8 & -\ 7x & +\ 0x^2 & +\ x^3 & x = \dfrac{1+x_1}{1} \\
 & +\ 8 & -\ 7 & +\ 0 & +\ 1 & \\
 & +\ 8 & -\ 7 & +\ 1 & +\ 1 & \\
 & +\ 8 & -\ 6 & +\ 2 & +\ 1 & \\
0 = & +\ 2 & -\ 4x_1 & +\ 3x_1{}^2 & +\ x_1{}^3 & x_1 = \dfrac{1+x_2}{2} \\
 & +\ 4 & -\ 8 & +\ 6 & +\ 1 & \\
 & +\ 8 & -\ 16 & +\ 7 & +\ 1 & \\
 & +\ 16 & -\ 9 & +\ 8 & +\ 1 & \\
0 = & +\ 7 & -\ 1x_2 & +\ 9x_2{}^2 & +\ x_2{}^3 & x_2 = \dfrac{1+x_3}{1} \\
 & +\ 7 & -\ 1 & +\ 9 & +\ 1 & \\
 & +\ 7 & -\ 1 & +\ 10 & +\ 1 & \\
 & +\ 7 & +\ 9 & +\ 11 & +\ 1 & \\
0 = & +\ 16 & +\ 20x_3 & +\ 12x_3{}^2 & +\ x_3{}^3 & x_3 = \dfrac{1+x_4}{-2} \\
 & -\ 32 & -\ 40 & -\ 24 & +\ 1 & \\
 & +\ 64 & +\ 80 & -\ 23 & +\ 1 & \\
 & -\ 128 & +\ 57 & -\ 22 & +\ 1 & \\
0 = & -\ 71 & +\ 35x_4 & -\ 21x_4{}^2 & +\ x_4{}^3 & x_4 = \dfrac{1+x_5}{1} \\
 & -\ 71 & +\ 35 & -\ 21 & +\ 1 & \\
 & -\ 71 & +\ 35 & -\ 20 & +\ 1 & \\
 & -\ 71 & +\ 15 & -\ 19 & +\ 1 & \\
0 = & -\ 56 & -\ 4x_5 & -\ 18x_5{}^2 & +\ x_5{}^3 & x_5 = \dfrac{1+x_6}{-1} \\
 & +\ 56 & +\ 4 & +\ 18 & +\ 1 & \\
 & -\ 56 & -\ 4 & +\ 19 & +\ 1 & \\
 & +\ 56 & +\ 15 & +\ 20 & +\ 1 & \\
0 = & +\ 71 & +\ 35x_6 & +\ 21x_6{}^2 & +\ x_6{}^3 & x_6 = \dfrac{1+x_7}{-1}
\end{array}
$$

We have, therefore, the periodic form

$$x = \frac{1}{1}\Big[1+\frac{1}{2}\Big[1+\frac{1}{1}\Big[1-\frac{1}{2}\Big[1+\frac{1}{1}\Big[1-\frac{1}{1}\Big[1-\frac{1}{1}\Big[1+\frac{1}{1}\Big[1-\frac{1}{1}\Big[1-\frac{1}{1}\Big[1+\ldots,$$

which obviously has no value.

The application of the method to the general quadratic

$$0 = a'' + a'x + x^2$$

is of great interest. Assuming $x = \dfrac{1+x_1}{-a'a''^{-1}}$, we have the transformed equation in x_1

$$0 = 1 + (2 - a'^2 a''^{-1})\, x_1 + x_1{}^2.$$

Assuming

$$x_1 = \frac{1 + x_2}{a'^2 a''^{-1} - 2},$$

we have the equation in x_2

$$0 = 1 + [2 - (a'^2 a''^{-1} - 2)^2]\, x_2 + x_2{}^2.$$

We have, therefore,

$$x = -\frac{1}{a'a''^{-1}}\left[1 + \frac{1}{a'^2 a''^{-1} - 2}\left[1 + \frac{1}{(a'^2 a''^{-1} - 2)^2 - 2}\left[1 \right.\right.\right.$$
$$\left. + \frac{1}{[(a'^2 a''^{-1} - 2)^2 - 2]^2 - 2}\left[1 + \ldots, \right.\right. \qquad (10)$$

or if we place

$$q = -a'a''^{-1},$$
$$q_1 = -a'^2 a''^{-1} - 2,$$

then will

$$q_2 = q_1{}^2 - 2,$$
$$q_3 = q_2{}^2 - 2, \qquad (11)$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

We have

$$x = -\frac{a'}{2} + \sqrt{\frac{a'^2}{4} - a''};$$

therefore,

$$\sqrt{\frac{a'^2}{4} - a''} = \frac{a'}{2} - \frac{a''}{a'}\left[1 + \frac{1}{a'^2 a''^{-1} - 2}\left[1 + \frac{1}{q_1{}^2 - 2}\left[1 + \frac{1}{q_2{}^2 - 2}\left[1 + \ldots \right.\right.\right.\right. \qquad (12)$$

Placing

$$z = -\frac{4a''}{a'^2}, \qquad (13)$$

and dividing by $\frac{a'}{2}$,

$$\sqrt{1 + z} = 1 + \frac{z}{2}\left[1 - \frac{1}{4z^{-1} + 2}\left[1 + \frac{1}{16z^{-2} + 16z^{-1} + 2}\left[1 + \ldots \right.\right.\right. \qquad (14)$$

Now this form, which is essentially Robert's rule for extracting a square root,* has the remarkable property of being *convergent for any value of z*, while its value by the binomial series

$$\sqrt{1 + z} = 1 + \frac{1}{2}z\left[1 - \frac{1}{4}z\left[1 - \frac{3}{6}z\left[1 - \frac{5}{8}z\left[1 - \ldots \right.\right.\right.\right.$$

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ceases to be convergent if $z > 1$; for the denominators q, q_1, q_2, . . . can never be < 2, whatever the value of z. If, however, $z < 1$, these denominators tend to infinity, which will bring about perfect identity, while the form is only convergent if $z > 1$.

If we assume

$$f(x) = \frac{1}{q}\Big[1 + \frac{1}{q_1 x^{-1} + q_1{}'}\Big[1 + \frac{1}{q_2 x^{-2} + q_2{}' x^{-1} + q_2{}''}\Big[1 + \frac{1}{q_3 x^{-3} + q_3{}' x^{-2} + q_3{}'' x^{-1} + q_3{}'''}\Big[1 + \ldots, \tag{15}$$

we have a form which becomes perfectly identical if $x < 1$, and is convergent only if $x > 1$, provided the quantities q, $q_1{}'$, $q_2{}''$, $q_3{}'''$, . . . tend to values > 1.

We may assume identically

$$f(x) = \frac{f(x)}{1 + f_1(x)}\Big[1 + \frac{f_1(x)}{1 + f_2(x)}\Big[1 + \frac{f_2(x)}{1 + f_3(x)}\Big[1 + \ldots, \tag{16}$$

so that

$$\begin{aligned} q &= \frac{1 + f_1(x)}{f(x)}, \\ q_1 x^{-1} + q_1{}' &= \frac{1 + f_2(x)}{f_1(x)}, \\ q_2 x^{-2} + q_2{}' x^{-1} + q_2{}'' &= \frac{1 + f_3(x)}{f_2(x)}, \\ q_3 x^{-3} + q_3{}' x^{-2} + q_3{}'' x^{-1} + q_3{}''' &= \frac{1 + f_4(x)}{f_3(x)}, \\ &\ldots\ldots\ldots \end{aligned} \tag{17}$$

and by McLaurin's theorem we have

$$\begin{aligned} f_1(x) &= f_1(0) + f_1{}'(0) \cdot x + \frac{f_1{}''(0)}{2!} x^2 + \ldots \\ &= -1 + q\Big[f(0) + f'(0) \cdot x + \frac{f''(0)}{2!} x^2 + \ldots\Big]; \end{aligned}$$

hence, if $f_1(0) = 0$,

$$0 = f_1(0) = -1 + qf(0); \quad \therefore q = \frac{1}{f(0)}, \tag{18}$$

$$\begin{aligned} f_1{}'(0) &= qf'(0), \\ \frac{f_1{}''(0)}{2!} &= q\frac{f''(0)}{2!}, \\ &\ldots\ldots \end{aligned} \tag{19$_1$}$$

We have further, if $f_2(0) = f_2'(0) = 0$,

$$f_2(x) = \frac{f_2''(0)}{2!}x^2 + \frac{f_2'''(0)}{3!}x^3 + \dots$$
$$= -1 + (q_1 + q_1'x)\left[f_1'(0) + \frac{f_1''(0)}{2!}x + \frac{f_1''(0)}{3!}x^2 + \dots\right];$$

hence

$$0 = f_2(0) = -1 + q_1 f_1'(0); \quad \therefore q_1 = \frac{1}{f_1'(0)}, \qquad (18_1)$$

$$0 = f_2'(0) = q_1\frac{f_1'(0)}{2!} + q_1' f_1'(0); \quad \therefore q_1' = -\frac{1}{f_1'(0)}q_1\frac{f_1''(0)}{2!}, \qquad (18_1')$$

$$\frac{f_2''(0)}{2!} = q_1\frac{f_1'''(0)}{3!} + q_1'\frac{f_1''(0)}{2!},$$

$$\frac{f_2'''(0)}{3!} = q_1\frac{f_1^{\text{iv}}(0)}{4!} + q_1'\frac{f_1'''(0)}{3!}, \qquad (19_2)$$

$$\dots \dots \dots \dots \dots \dots \dots$$

$$0 = f_3(0) = -1 + q_2\frac{f_2''(0)}{2!}; \quad \therefore q_2 = \frac{2!}{f_2''(0)}, \qquad (18_2)$$

$$0 = f_3'(0) = q_2\frac{f_2'''(0)}{3!} + q_2'\frac{f_2''(0)}{2!}; \quad \therefore q_2' = -\frac{2!}{f_2''(0)}q_2\frac{f_2'''(0)}{3!}, \qquad (18_2')$$

$$0 = \frac{f_3''(0)}{2!} = q_2\frac{f_2^{\text{iv}}(0)}{4!} + q_2'\frac{f_2'''(0)}{3!} + q_2''\frac{f_2''(0)}{2!};$$

$$\therefore q_2'' = -\frac{2!}{f_2''(0)}\left[q_2\frac{f_2^{\text{iv}}(0)}{4!} + q_2'\frac{f_2'''(0)}{3!}\right], \qquad (18_2'')$$

$$\frac{f_3'''(0)}{3!} = q_2\frac{f_2^{\text{v}}(0)}{5!} + q_2'\frac{f_2^{\text{iv}}(0)}{4!} + q_2''\frac{f_2'''(0)}{3!},$$

$$\frac{f_3^{\text{iv}}(0)}{4!} = q_2\frac{f_2^{\text{vi}}(0)}{6!} + q_2'\frac{f_2^{\text{v}}(0)}{5!} + q_2''\frac{f_2^{\text{iv}}(0)}{4!}, \qquad (19_3)$$

$$\dots \dots \dots \dots \dots \dots \dots$$

and so on.

If the proposed function $f(x)$ vanishes for $x = 0$ by virtue of a factor x^p (where p may be either whole or fractional), we apply the above process to the function

$$x^{-p}f(x), \text{ denoting it } f(x),$$

or to

$$x^{-p+1}f(x), \quad \text{"} \quad \text{"} \quad f_1(x),$$
$$x^{-p+2}f(x), \quad \text{"} \quad \text{"} \quad f_2(x),$$
$$\dots \dots \dots \dots$$

If the function is par, or $f(x) = f(-x)$, its development contains only even powers of x, and placing $x^2 = z$ we have the required form. If the

function is impar, or $f(x) = -f(-x)$, dividing by x and placing $x^2 = z$ we have again the required form.

We may also arbitrarily modify the complete form (15), if, for instance, we do not determine all the q's at the nth term. This, however, changes nothing essential in the rule in equations (18) and (19). At any step there will be as many equations (18) as we choose; then formulæ (19) follow where (18) have been interrupted. To explain this modification, let us suppose we only want two q's in the fourth term, then we have

$$0 = f_4\ (0) = -1 + q_3 \frac{f_3'''(0)}{3\,!}; \quad \therefore q_3 = \frac{3\,!}{f_3'''(0)}, \tag{18_3}$$

$$0 = f_4'\ (0) = q_3 \frac{f_3^{\text{iv}}(0)}{4\,!} + q_3' \frac{f_3'''(0)}{3\,!}; \quad \therefore q_3' = -\frac{3\,!}{f_3'''(0)} q_3 \frac{f_3^{\text{iv}}(0)}{4\,!}, \tag{$18_3'$}$$

$$\frac{f_4''(0)}{2\,!} = q_3 \frac{f_3^{\text{v}}(0)}{5\,!} + q_3' \frac{f_3^{\text{iv}}(0)}{4\,!},$$

$$\frac{f_4'''(0)}{3\,!} = q_3 \frac{f_3^{\text{vi}}(0)}{6\,!} + q_3' \frac{f_3^{\text{v}}(0)}{5\,!}, \tag{19}$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

It happens in some cases, much to the advantage of the form (15), that one or more of equations (19) vanish. In that case a corresponding number of terms in (15) also disappear.

For the first application take the function

$$f(x) = l(1 + x).$$

Since here

$$f(0) = 0 \qquad \text{and} \qquad f'(0) = 1,$$

we put

$$f_1(x) = l(1 + x),$$

so that

$$f_1\ (0) = 0,$$

$$f_1'\ (0) = 1,$$

$$\frac{f_1''(0)}{2\,!} = -\frac{1}{2},$$

$$\frac{f_1'''(0)}{3\,!} = +\frac{1}{3},$$

$$\cdot \quad \cdot \quad \cdot$$

$$\frac{f_1^{(m)}(0)}{m\,!} = (-)^{m-1} \frac{1}{m}.$$

We have then

$$0 = q_1 f_1'(0) - 1\,; \quad \therefore q_1 = 1,$$

$$0 = q_1 \frac{f_1''(0)}{2!} + q_1' f_1'(0)\,; \quad \therefore q_1' = \frac{1}{2},$$

$$\frac{f_2''(0)}{2!} = 1 \times + \frac{1}{3} + \frac{1}{2} \times - \frac{1}{2} = \frac{1}{3 \,.\, 2^2} = \frac{1}{12},$$

$$\frac{f_2'''(0)}{3!} = 1 \times - \frac{1}{4} + \frac{1}{2} \times + \frac{1}{3} = \frac{-2}{4 \,.\, 2 \,.\, 3} = - \frac{1}{12},$$

$$\frac{f_2^{\text{iv}}(0)}{4!} = 1 \times + \frac{1}{5} + \frac{1}{2} \times - \frac{1}{4} = \frac{3}{5 \,.\, 2 \,.\, 4} = \frac{3}{40},$$

. . . .

$$\frac{f_2^{(m)}(0)}{m!} = \left[1 \times \frac{1}{m} + \frac{1}{2} \times - \frac{1}{m-1}\right](-)^{m-1} = \frac{m-1}{2m\,(m-1)}\,(-)^{m-1}\,;$$

and

$$0 = q_2 \frac{f_2''(0)}{2!} - 1\,; \quad \therefore q_2 = 12,$$

$$0 = q_2 \frac{f_2'''(0)}{3!} + q_2' \frac{f_2''(0)}{2!}\,; \quad \therefore q_2' = 12,$$

$$0 = q_2 \frac{f_2^{\text{iv}}(0)}{4!} + q_2' \frac{f_2'''(0)}{3!} + q_2'' \frac{f_2''(0)}{2!}\,; \quad \therefore q_2'' = \frac{6}{5},$$

$$\frac{f_3'''(0)}{3!} = 12 \times \frac{-4}{2 \,.\, 5 \,.\, 6} + 12 \times \frac{3}{2 \,.\, 4 \,.\, 5} + \frac{6}{5} \times \frac{-2}{2 \,.\, 3 \,.\, 4} = 0,$$

$$\frac{f_3^{\text{iv}}(0)}{4!} = 12 \times \frac{5}{2 \,.\, 6 \,.\, 7} + 12 \times \frac{-4}{2 \,.\, 5 \,.\, 6} + \frac{6}{5} \times \frac{3}{2 \,.\, 4 \,.\, 5} = \frac{3}{700},$$

$$\frac{f_3^{\text{v}}(0)}{5!} = 12 \times \frac{-6}{2 \,.\, 7 \,.\, 8} + 12 \times \frac{5}{2 \,.\, 6 \,.\, 7} + \frac{6}{5} \times \frac{-4}{2 \,.\, 5 \,.\, 6} = \frac{-3}{350},$$

$$\frac{f_3^{\text{vi}}(0)}{6!} = 12 \times \frac{7}{2 \,.\, 8 \,.\, 9} + 12 \times \frac{-6}{2 \,.\, 7 \,.\, 8} + \frac{6}{5} \times \frac{5}{2 \,.\, 6 \,.\, 7} = \frac{1}{84},$$

$$\frac{f_3^{\text{vii}}(0)}{7!} = 12 \times \frac{-8}{2 \,.\, 9 \,.\, 10} + 12 \times \frac{7}{2 \,.\, 8 \,.\, 9} + \frac{6}{5} \times \frac{-5}{2 \,.\, 7 \,.\, 8} = \frac{1}{70},$$

$$\frac{f_3^{\text{viii}}(0)}{8!} = 12 \times \frac{9}{2 \,.\, 10 \,.\, 11} + 12 \times \frac{-8}{2 \,.\, 9 \,.\, 10} + \frac{6}{5} \times \frac{7}{2 \,.\, 8 \,.\, 9} = \frac{7}{440},$$

$$\frac{f_3^{\text{ix}}(0)}{9!} = 12 \times \frac{-10}{2 \,.\, 11 \,.\, 12} + 12 \times \frac{9}{2 \,.\, 10 \,.\, 11} + \frac{6}{5} \times \frac{-8}{2 \,.\, 9 \,.\, 10} = - \frac{14}{825},$$

$$\frac{f_3^{\text{x}}(0)}{10!} = 12 \times \frac{11}{2 \,.\, 12 \,.\, 13} + 12 \times \frac{-10}{2 \,.\, 11 \,.\, 12} + \frac{6}{5} \times \frac{9}{2 \,.\, 10 \,.\, 11} = \frac{63}{3575},$$

. . . .

Since here $\frac{f_3'''(0)}{3!}$ vanishes, the third term then disappears, and we have

$$0=\frac{3}{700}q_4-1;\quad \therefore q_4=\frac{700}{3},$$

$$0=-\frac{3}{350}q_4+\frac{3}{700}q_4';\quad \therefore q_4'=\frac{1400}{3},$$

$$0=\frac{1}{84}q_4-\frac{3}{350}q_4'+\frac{3}{700}q_4'';\quad \therefore q_4''=\frac{7700}{27},$$

$$0=-\frac{1}{70}q_4+\frac{1}{84}q_4'-\frac{3}{350}q_4''+\frac{3}{700}q_4''';\quad \therefore q_4'''=\frac{1400}{27},$$

$$0=\frac{7}{440}q_4-\frac{1}{70}q_4'+\frac{1}{84}q_4''-\frac{3}{350}q_4'''+\frac{3}{700}q_4^{\text{iv}};\quad \therefore q_4^{\text{iv}}=\frac{2450}{2673},$$

$$\frac{f_5^{\text{v}}(0)}{5!}=-\frac{14}{825}\cdot\frac{700}{3}+\frac{7}{440}\cdot\frac{1400}{3}-\frac{1}{70}\cdot\frac{7700}{27}+\frac{1}{84}\cdot\frac{1400}{27}-\frac{3}{350}\cdot\frac{2450}{2673}=0,$$

$$\frac{f_5^{\text{vi}}(0)}{6!}=\frac{63}{3575}\cdot\frac{700}{3}-\frac{14}{825}\cdot\frac{1400}{3}+\frac{7}{440}\cdot\frac{7700}{27}-\frac{1}{70}\cdot\frac{1400}{27}+\frac{1}{84}\cdot\frac{2450}{2673}=-\frac{10}{104247},$$

etc., etc.

Since here $q_1'=\frac{1}{2}$, $q_2''=\frac{6}{5}$, $q_4^{\text{iv}}=\frac{2450}{2673}$, . . . do not appear to tend to a value >1, we cannot assert that the form

$$l\,(1+x)=\frac{x}{1+\frac{1}{2}x}\left[1+\frac{x^2}{12+12x+\frac{6}{5}x^2}\left[1+\frac{x^4}{\frac{700}{3}+\frac{1400}{3}x+\frac{7700}{27}x^2+\frac{1400}{27}x^3+\frac{2450}{2673}x^4}\left[1-\ldots\right.\right.\right. \tag{20}$$

is convergent for every value of x. It may, however, be used for values of $x>1$, to a considerable extent, as long as the denominators q, $q_1x^{-1}+q_1'$, $q_2x^{-2}+q_2'x^{-1}+q_2''$, . . . form an ascending series. If $x<1$, the form is perfectly identical, and the first two terms will be correct to at least six decimal places.

Placing $x=\frac{z}{1-z}$, we have

$$l\,(1+x)=l\left[1+\frac{z}{1-z}\right]=-l\,(1-z);$$

and by (20),

$$l\,(1+x) = -\,l\,(1-z) = \frac{z}{1-\frac{1}{2}z}\left[1+\frac{z^2}{12-12z+\frac{6}{5}z^2}\left[1\right.\right.$$

$$\left.\left.+\frac{z^4}{\frac{700}{3}-\frac{1400}{3}z+\frac{7700}{27}z^2-\frac{1400}{27}z^3+\frac{2450}{2673}z^4}\left[1-\ldots\right.\right.\right. \quad (21)$$

Since $z = \frac{x}{1+x} < 1$, this form is always perfectly identical.

This example is a fair exhibit of the advantages and defects of the method. Although the required coefficients q are determined by a regular algorithmic process, the resulting form is by no means of such a form that it might be continued by induction, except in case of Robert's rule (14) and the formula for the root of a quadratic (10).